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Well-Set Cauchy Problems and C_0-Semigroups*

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I. INTRODUCTION

The purpose of this paper is to develop an idea of Hille [1, Chap. XX]: that well-set Cauchy problems with time-independent coefficients can be effectively represented by semigroups of bounded linear transformations on Banach spaces. His idea is justified, after being reinterpreted along lines already suggested in [2], for systems of *linear* partial differential equations with *constant coefficients*.

Any such system can be reduced to the *normal form*

$$\frac{\partial u_j}{\partial t} = \sum_{k=1}^{n} p_{jk}(D_1, \cdots, D_r)\, u_k, \qquad D_i = \frac{\partial}{\partial x_i}, \tag{1}$$

where $P = \| p_{jk} \|$ is an $n \times n$ matrix of polynomial functions with real or *complex* coefficients:

$$p_{jk}(\mathbf{D}) = \sum a_{jk\mathbf{l}}\, D_1^{l_1} \cdots D_r^{l_r}, \qquad \mathbf{l} = (l_1, \cdots, l_r). \tag{1'}$$

The components $u_j(\mathbf{x}, t)$ of the vector fields $\mathbf{u}(\mathbf{x}, t)$ are complex-valued functions of the $r + 1$ *real* variables $x_1, \cdots, x_r, t$.

Hille's basic idea was to construct a *Banach space* $\mathscr{B}$ for each well-set Cauchy problem (with time-independent coefficients), whose points represented possible $\mathbf{u}(\mathbf{x}, t)$ for *fixed* t, and to express the solution for given $\omega_0 = \mathbf{u}(\mathbf{x}, 0) \in \mathscr{B}$ as the orbit $\omega(t) = T_t[\omega_0] = \mathbf{u}(\mathbf{x}, t)$ emanating from ω_0 under the action of a suitable semigroup $\{T_t\}$ of bounded linear transformations on $\mathscr{B}$; see [1].

In developing this idea, the domain of $\mathbf{x}$ will be taken as an arbitrary product $X = R^s C^{r-s}$ of real straight lines R and circles C. It is well known[1] that such a product represents the most general locally Euclidean Abelian

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[1] See, for example L. S. Pontrjagin, "Topologische Gruppen," Vol. 2, p. 22. Teubner, Leipzig, 1958.

group manifold, and has a dual character group $X^* = Q = R^s Z^{r-s}$, where Z is the additive group of integers. Hence, to specify the system (1) in the large, one must specify s as well as r, n, and finitely many nonzero coefficients $a_{jk\mathbf{l}}$.

Most of the methods used in the present paper were already sketched in [2]. But a more thorough study of the Cauchy problem (1) made it clear to the author that these methods needed to be developed and clarified further. In particular, their present form will be used to establish the convergence of *difference approximations* to (1) on uniform meshes H (discrete subgroups of X), in a paper being written in collaboration with Prof. Richard S. Varga.[2]

II. Fourier Transforms

Fourier transforms have been used to solve Cauchy problems for systems of the form (1) for a long time.[3] Typically, one first writes down Fourier transforms formally, and then (often much later) verifies *a posteriori* that the formulas give rigorously correct results.

Formally [3, Chap. III, § 5], one defines the multiple Fourier transform by

$$\mathbf{f}(\mathbf{q}, t) = \int_{\Omega} \mathbf{u}(\mathbf{x}, t)\, e^{i\mathbf{q}\cdot\mathbf{x}}\, dX, \qquad dX = dx_1 \cdots dx_r\,, \tag{2}$$

the inverse transformation being

$$\mathbf{u}(\mathbf{x}, t) = (2\pi)^{-r} \int_{\Omega^*} \mathbf{f}(\mathbf{q}, t)\, e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ, \qquad dQ = dq_1 \cdots dq_r\,. \tag{2$'$}$$

Then, formally, $\mathbf{u}(\mathbf{x}, t)$ satisfies (1) if and only if $\mathbf{f}(\mathbf{q}, t)$ satisfies for each *real* wave vector $\mathbf{q}$ the system of linear *ordinary* differential equations

$$\frac{df_j}{dt} = \sum_{k=1}^{n} p_{jk}(iq_1\,, \cdots, iq_r)\, f_k\,, \tag{3}$$

which also has constant coefficients. The solution of (3) can be expressed very simply in matrix notation, as

$$\mathbf{f}(\mathbf{q}, t) = \exp\,(tP(i\mathbf{q}))\, \mathbf{f}(\mathbf{q}, 0). \tag{3$'$}$$

Much effort has gone into developing rigorous interpretations of the preceding formulas, for whose validity Fourier made extravagant claims. The most striking interpretation is given by Plancherel's Theorem, which says

[2] The author also wishes to thank Prof. Varga for his partial collaboration on the present paper, and Drs. Robert E. Lynch and Thomas Mullikin for helpful criticisms.

[3] Beginning with A. Cauchy, *J. École Polytech.* **12**, 511 (1823).

that (up to a constant norm factor) formulas (2) and (2′) define reciprocal *isometries* between the Hilbert spaces $L_2(X)$ and $L_2(X^*)$. Somewhat more generally, (2′) maps $L_p(X^*)$ (for $1 \leqslant p \leqslant 2$) onto $L_{p'}(X)$, where $p' = p/(p-1)$. In particular, it maps $L_1(X^*)$ into $C(X) \subset L_\infty(X)$.

Recently, more general interpretations have been given in terms of Schwartz distributions; see Section VI.

III. Regularity

Measures of the *stability* of systems (1) were developed in the 19th century by Kelvin, Rayleigh, and other mathematical physicists, again using Fourier analysis nonrigorously. Generalizing their ideas to the complex domain, one can define a *normal mode* with *wavevector* $\mathbf{q} \in X^*$ for the system (1) as a vector field $\mathbf{f}e^{i\mathbf{q}\cdot\mathbf{x}}$ such that $P(i\mathbf{q})\,\mathbf{f} = \lambda(i\mathbf{q})\,\mathbf{f}$; the number $\lambda(i\mathbf{q})$ is an *eigenvalue* of $P(i\mathbf{q})$ (and the mode). For any fixed $\mathbf{q} \in X^*$, these eigenvalues are clearly the zeros of the *stability polynomial* of the system (1), defined as in [3]

$$| P(i\mathbf{q}) - \lambda I | = \prod_{l=1}^{n} [\lambda_l(\mathbf{q}) - \lambda]. \tag{4}$$

(The equation $| P(i\mathbf{q}) - \lambda I | = 0$ is called the *characteristic equation* of (1).)

Let the *algebraic spectrum* of the system (1) on X be defined as the set of *all* eigenvalues $\lambda_l(\mathbf{q})$ of $P(i\mathbf{q})$, as $\mathbf{q}$ ranges over X^*. Except in degenerate cases, this algebraic spectrum is discrete if and only if X is compact.

The system (1) will be called *strictly stable* on X if and only if *all* $\lambda_l(i\mathbf{q})$ have negative real parts; this condition defines *dissipative* systems. We now make an even more important definition.

Definition. The *stability index* $\Lambda(P, X)$ of the system (1) on X is $\sup_{X^*,l} \operatorname{Re}\{\lambda_l(P(iq))\}$. The system (1) is *regular* on X if and only if $\Lambda(P, X) < +\infty$.

If the system (1) is strictly stable on X, then $\Lambda(P, X) \leqslant 0$. The concept of the stability index, and its importance, were certainly recognized by Kelvin and Rayleigh. In various problems, they determined the "least stable" (or "most unstable") wavelength $2\pi/q_m$ for which some $\lambda_l(q_m) = \Lambda$, and the associated normal mode.

The preceding concepts are illustrated in the following two examples.

Example 1. Consider the (parabolic) complex heat equation $u_t = \alpha u_{xx}$, where α is an arbitrary real or complex number, on $R = X_{1,1}$. In this example, (3′) gives

$$f(q, t) = f(q, 0)\, e^{-\alpha q^2 t}. \tag{5}$$

Hence $u_t = \alpha u_{xx}$ defines a semigroup of linear contractions (norm-diminishing transformations) on $L_2(X)$ provided $\mathrm{Re}\{\alpha\} \geqslant 0$. It does not define a semigroup of bounded linear transformations on $L_2(X)$ if $\mathrm{Re}\{\alpha\} < 0$.

Example 2. Consider the complex convection equation $u_t = \alpha u_x$ on R. Here (3′) gives

$$f(q, t) = f(q, 0)\, e^{i\alpha q t}. \tag{6}$$

In this example, if α is real, $u_t = \alpha u_x$ defines a semigroup of isometries (norm-preserving transformations) on $L_2(X)$: this is the ***hyperbolic*** case [3, p. 216]. But if α is complex, then the ratios $|f(q, t)|/|f(q, 0)|$ form an unbounded set for all $t \neq 0$, and one has the same difficulty as before.

The following examples show that (strict) stability and regularity[4] depend not only on the coefficients of (1), but also on the choice of X.

Example 3. Consider the parabolic DE

$$u_t + u + 5u_{xx} + 5u_{xxxx} = 0, \qquad n = r = 1. \tag{7}$$

One easily computes $\lambda(q) = -1 + 5q^2 - 5q^4$, whence $\lambda(1/\sqrt{2}) = 1/4$; hence (7) is not stable on the real line. But on the circle $0 \leqslant x \leqslant 2\pi$, it is strictly stable since $\lambda(N) = -1 + 5N^2(1 - N^2) < 0$ for any integer N.

Example 3′. Consider the parabolic DE

$$u_t = u_{xxxxyy} + 2u_{xxxy}\,. \tag{8}$$

Then

$$P(i\mathbf{q}) = -\,q_1^4 q_2^2 + 2q_1^3 q_2 = q_1^2[1 - (q_1 q_2 - 1)^2].$$

This is unbounded on the hyperbola $q_1 q_2 = 1$; hence it is irregular on $X_{2,2}$. But on the network of wavevectors $(m_1, m_2) \in X_{2,0}^*$ with integral coordinates, $P(i\mathbf{q}) \leqslant 0$ since $(q_1 q_2 - 1)^2 \geqslant 1$. Hence (8) is regular on $X_{2,0}$.

Moreover every direction is regular in Example 3′, in the sense of the following definition.

DEFINITION. A direction $\gamma \neq \mathbf{0}$ is *regular* when

$$\sup_{l,s} \mathrm{Re}\,\{\lambda_l(P(is\gamma))\} < +\infty \qquad (s \text{ real}),$$

and *irregular* otherwise.

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[4] The author is much indebted to Prof. A. Seidenberg for suggesting Example 3′, for proving the lemma below, and for other important observations concerning the notion of "local irregularity."

Example 3''. Consider the parabolic *DE*

$$u_t = -u - u_{xx} + u_{xxxxxx} - 4u_{xxxxyy} + 4u_{xxyyyy}. \tag{8'}$$

Then

$$P(i\mathbf{q}) = -1 + q_1^2[1 - (q_1^2 - 2q_2^2)^2].$$

For this *DE*, the directions $(\pm\sqrt{2}, 1)$ are irregular; hence the *DE* is irregular. But all other directions are regular; hence the set of regular directions is not closed. Finally, since $P(i\mathbf{q}) \leqslant -1$ on the network of wave vectors with integral coordinates, (8') is regular and even strictly stable on $X_{2,0}$.

The significance of the preceding examples is, of course, that the notion of the regularity of a system (1) does not depend on its coefficients alone. To express this idea concisely, a new notion of local irregularity will be defined, as follows.

DEFINITION. The system (1) is *locally irregular* if it is irregular on every locally Euclidean r-parameter Abelian group X.

This is equivalent to the condition that, for any r linearly independent vectors $\mathbf{h}_1, \cdots, \mathbf{h}_r$ in R^r,

$$\sup_{\mathbf{m},l} \operatorname{Re}\{\lambda_l(P(im_1\mathbf{h}_1 + \cdots + im_r\mathbf{h}_r))\} = +\infty,$$

where $m_1, \cdots, m_r$ are arbitrary integers and $l = 1, \cdots, r$.

Examples 3' and 3'' are systems (1) with $n = 1$ and $r = 2$, which are irregular in R^r without being locally irregular. That this situation is exceptional is illustrated by the following results.

THEOREM 1. *If $r = 1$, or if $l_1 + \cdots + l_r = \nu$ is constant in (1), then any system (1) is either regular for all $X_{r,s}$ or irregular for all $X_{r,s}$. In other words, if (1) is irregular then it is locally irregular.*

For $r = 1$, this result was essentially proved by A. Lax[5], using Puiseux's Theorem as a tool. The converse is trivial.

COROLLARY. *If, for the set Z of all integers,* $\sup_{l,m\in Z} \operatorname{Re}\{\lambda_l(P(im\gamma))\} < +\infty$, *then γ is a regular direction.*

For the case $l_1 + \cdots + l_r = \nu$ of a "homogeneous" system (1), Theorem 1 is a consequence of the following new result.

[5] A. Lax, *Commun. Pure Appl. Math.* **9**, 135-69 (1956); see also R. Courant and A. Lax, *ibid.* **8**, 497-502 (1955). For Puiseux's Theorem, see C. Jordan, "Cours d'Analyse," vol. 1, § 361. For other applications of Puiseux's Theorem, see E. A. Gorin, *Usp. Mat. Nauk SSSR* **16**, 91-118 (1961) and [4, pp. 219-35].

LEMMA (A. Seidenberg). *Let $P = (p_{jk})$ be an $n \times n$ matrix of polynomials $P_{jk} = p_{jk}(q_1, \cdots, q_r)$ with real or complex coefficients, and assume that the p_{jk} are homogeneous, of degree ν. Let $\lambda_1, \cdots, \lambda_n$ be the characteristic roots of P and assume that* $\sup \operatorname{Re}\{\lambda_l(\mathbf{q})\} = +\infty$, *where l varies from 1 to n and $\mathbf{q}$ varies over real affine r-space R^r. Then* $\sup \operatorname{Re}\{\lambda_l(\mathbf{q})\} = +\infty$ *also when $\mathbf{q}$ varies over the lattice points of R^r.*

PROOF. Let $\mathbf{q}$ be a point of R^r other than the origin $\mathbf{0}$, and let the ray from $\mathbf{0}$ to $\mathbf{q}$ cut the unit sphere S in γ, so that $\mathbf{q} = q\gamma$, $q > 0$. Considering $| P(\mathbf{q}) - \lambda I | = q^{\nu n} | P(\gamma) - \lambda q^{-\nu} I |$, one sees that the characteristic roots of $P(\mathbf{q})$ are q^ν times those of $P(\gamma)$. Let $\bar{\lambda}_1, \cdots, \bar{\lambda}_n$ be the characteristic roots of $P(\gamma)$. If the real part of each root is nonpositive for every $\gamma \in S$, then the same is true of $\lambda_1, \cdots, \lambda_n$ for every $\mathbf{q} \in R^r$, and so $\sup \operatorname{Re}\{\lambda_l(\mathbf{q})\} \leqslant 0$. On the other hand, if some $\bar{\lambda}_l$ has a positive real part, then the direction from 0 to γ is irregular, and $\sup \operatorname{Re}\{\lambda_l(\mathbf{q})\} = +\infty$. Thus the hypothesis of the lemma holds if and only if some $\bar{\lambda}_l$ has a positive real part for some $\gamma \in S$.

Let γ_0 be a point of S for which $P(\gamma_0)$ has a characteristic root λ_0 having a positive real part. Let $\lambda_1, \cdots, \lambda_n$ be the roots of $| P(\mathbf{q}) - \lambda I | = 0$, so that $\Pi(\lambda_l - \lambda) = | P(\mathbf{q}) - \lambda I |$. Then $\Pi(\lambda_l - \lambda_0) = | P(\mathbf{q}) - \lambda_0 I |$, and this approaches $| P(\gamma) - \lambda_0 I |$, which is zero, as $\mathbf{q} \to \gamma_0$. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\Pi | \lambda_i(\mathbf{q}) - \lambda_0 | < \epsilon^n$ if $| \mathbf{q} - \gamma_0 | < \delta$. This implies that $| \lambda_i - \lambda_0 | < \epsilon$ for at least one i; and if ϵ is small enough, then λ_i will also have a positive real part. Thus there is a neighborhood U of γ_0 on S such that for any $\mathbf{q} \in U$, at least one of the characteristic roots of $P(\mathbf{q})$ has a positive real part (whence the direction from 0 to $\mathbf{q}$ is irregular).

Thus every direction in the cone projecting U from O is irregular. This cone contains a lattice point $\mathbf{m} \neq (0, \cdots, 0)$. Then $\sup \operatorname{Re}\{\lambda_l(kz)\} = +\infty$, where $k = 1, 2, \cdots$. This completes the proof.

IV. WELL-SET CAUCHY PROBLEMS

Following Hadamard, most mathematicians would agree that a Cauchy problem such as (1) should be called *well-set* when the solution at time t exists and is unique for given initial $\mathbf{u}(\mathbf{x}, 0)$, *and* makes $\mathbf{u}(\mathbf{x}, t)$ depend continuously on $\mathbf{u}(\mathbf{x}, 0)$. Unfortunately, this answer is highly ambiguous, until one has specified the *class* of fuctions admitted, together with a *topology* on the space of all "admissible" functions.

In spite of this ambiguity, various interpretations support the conclusion that the Cauchy problem for (1) should be considered as *well-set* (properly posed) if and only if (1) is *regular* (see [4, p. 198]). This conclusion was essentially reached by Hadamard, and arguments supporting it have been given by Petrowsky, Gårding, Hörmander, and others [5; 6; 4, pp. 330-1; 7]. For the backwards heat equation (Example 1 with $A < 0$), it was already reached by Maxwell[6].

I will follow Hille [1, Chap. XX] in requiring that $\mathbf{u}(\mathbf{x}, t) = T_t[\mathbf{u}(\mathbf{x}, 0)]$ should belong to a fixed Banach space $\mathscr{B}$ for each fixed $t \geqslant 0$, where $\{T_t\}$ is a *semigroup* of bounded linear transformations.

As was observed in [1, Chap. XX], the notion of semigroup of bounded linear transformations is directly related to Hadamard's ideas about well-set problems. In such a semigroup, there is one and only one orbit $w(t)$ for each initial "state" $w(0) = w_0$. Moreover $w(t)$ depends continuously on the initial "state" $w(0) = w_0$ and t; and $w(t)$ satisfies Huyghens' major premise (see [8, § 33] and [3, p. 215]).

I will further follow Hille-Phillips [9] in requiring that the semigroup $\{T_t\}$ be a C_0-semigroup in the sense of the usual definition[7].

If $X = X_{r,0} = C^r$ is *compact* (a torus), as was assumed for simplicity in [10] and [11], then one can show that regularity is *necessary* for the system (1) to define a C_0-semigroup in any reasonable sense. For, any reasonable Banach space $\mathscr{B}$ of "states" must contain the (bounded, integrable, analytic) functions $\mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}$ for any fixed n-vector $\mathbf{b}$ and wave-vector $\mathbf{q} \in X^* = Q$. On the other hand, the set $\mathscr{G}(\mathbf{q})$ of all such functions is a subspace invariant under the semigroup (3′). The norm of any T_t is at least the spectral radius of the matrix $e^{tP(i\mathbf{q})}$, which expresses the effect of T_t on $\mathscr{G}(\mathbf{q})$. Hence (1) cannot define a C_0-semigroup for compact X, unless the matrices $e^{tP(i\mathbf{q})}$ have *uniformly bounded spectral radii.*

But this is precisely the condition $\Lambda(P) < +\infty$ that (1) be regular, proving the following result.

THEOREM 2. *Unless (1) is regular, it cannot be represented as a C_0-semigroup on any reasonable Banach space of functions on C^r.*

It would be desirable to have a rigorous extension of the preceding result to noncompact X; the conclusion is very plausible.

Boundary Conditions

The rigorous analysis of *mixed* initial value problems involving spatial boundary conditions is also technically difficult. An exception is provided by the boundary conditions $u = 0$ and $\partial u/\partial n = 0$ on products of intervals, which can be associated with periodic continuations on products of circles.

Thus, one cannot simply restrict attention to the subspace of functions

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[6] J. C. Maxwell, "Theory of heat," 10th ed., pp. 264-5, 1891.

[7] [9, pp. 321, 359]. The class of semigroups being considered was not specified in [1], [10], or [11].

satisfying these conditions, as suggested in [11, p. 38], because the subspace of such functions is almost never closed (a Banach space). For correct treatments of spatial boundary conditions see, for example, refs. 12 and 13.

V. Examples of C_0-Semigroups

The analysis of Examples 1 and 2 can be extended to the general case $n = 1$. In this case, the system (1) assumes the simplified form

$$\frac{\partial u}{\partial t} = p(\mathbf{D})\, u = \sum_{\mathbf{l}} a_{\mathbf{l}} \frac{\partial^{l_1 + \cdots + l_r} u}{\partial x_1^{l_1} \cdots \partial x_r^{l_r}}, \qquad \mathbf{l} = (l_1, \cdots, l_r). \tag{10}$$

For any *regular DE* (10), one can realize the Cauchy problem by a C_0-semigroup on the Hilbert space $L_2(X) = \mathscr{B}$ by the following well-known construction (cf. [10, § 10] and [11, Chap. IV, § 3]).

By the Plancherel Theorem, formulas (2) and (2′) define an *isometry* $L_2(X) \cong L_2(X^*)$ (apart from a change of scale). Moreover (3) gives $f(\mathbf{q}, t) = e^{p(i\mathbf{q})t} f(\mathbf{q}, 0)$. Therefore

$$\| u(\mathbf{x}, t) \| = [(2\pi)^r \int_{\Omega} e^{2tp(i\mathbf{q})} \, | f(\mathbf{q}, 0) |^2 \, dQ]^{1/2}, \tag{11}$$

whence an elementary calculation gives

$$\| u(\mathbf{x}, t) \| \leqslant e^{t\Lambda} \, \| u(\mathbf{x}, 0) \|. \tag{11′}$$

It follows that the system (10) defines a C_0-semigroup on $L_2(X)$. Therefore, by Theorem 2, if $n = 1$, the system (1) defines an acceptable C_0-semigroup if and only if it is *regular*.

However, when $n > 1$, regular systems (1) do *not* always define C_0-semigroups on $L_2(X)$, as the following example[8] shows (cf. [9, Chap. VIII, § 2], and contrast with the assertions of [10, § 10] and [9, Chap. IV, § 3]).

Example 4. Consider the system $u_t = v$, $v_t = \nabla^2 u$ obtained from the wave equation $u_{tt} = \nabla^2 u$ by the standard recipe [4, p. 165, (1.6)]. The system is regular but, with the standard norm

$$\| (u, v) \| = \left[(2\pi)^r \int_X (u^2 + v^2) \, dX \right]^{1/2}$$

on $[L_2(X)]^2$, it does not define a C_0-semigroup.

[8] The somewhat more subtle example $u_t = v_{xxxx}$, $v_t = -u_{yyyy}$ was given on pp. 20-2 of my report LA-HU-2, Harvard University, September 1953. This can be realized by setting $\tilde{u} = u_{yy}$, $v = v_{xx}$, whence $\tilde{u}_t = v_{xxyy}$, $v_t = -\tilde{u}_{xxyy}$.

To construct a C_0-semigroup for $u_{tt} = \nabla^2 u$, one can use the alternative variables $v = u_t$, $u_j = \partial u/\partial x_j$, getting the system $\partial u_j/\partial t = \partial v/\partial x_j$, $\partial v/\partial t = \sum \partial u_k/\partial x_k$. This system defines a C_0-semigroup on $[L^2(X)]^{r+1}$, defined by the standard L_2-norm on the fields of $(r+1)$-vectors with values $(\partial u/\partial x_1, \cdots \partial u/\partial x_r, \partial u/\partial t)$; the norm[9] is then the square root of the physical "energy."

One can also realize the system $u_t = v$, $v_t = \nabla^2 u$, as a C_0-semigroup of isometries of the Hilbert space defined by the norm

$$\| (u, v) \| = \left\{ \int_{X^*} \left(\sum q_i^2 \right) | f |^2 + | g |^2 \, dQ \right\}^{1/2},$$

where $f(q)$ and $g(q)$ are the *Fourier transforms* of u and v, respectively. This is the construction which will be generalized below. The preceding example has many analogs, such as

Example 5. The vibrating beam equation $u_{tt} = -u_{xxxx}$ can be reduced to the form (1) by the usual prescription [10, p. 269, para. 3] of setting $v = u_t$ This gives $u_t = v$, $v_t = -u_{xxxx}$, whence $P(iq) = \left(\begin{smallmatrix} 0 & 1 \\ -q^4 & 0 \end{smallmatrix}\right)$. For all $t \neq 0$, the norm of the matrices $\exp(tP(iq))$ on the Euclidean plane is unbounded as $q \to \infty$; hence (1) fails to define a C_0-semigroup on $L_2(X)$.

On the other hand, as is well known, the *energy* norm

$$\| w \| = \left[\tfrac{1}{2} \int_X (v^2 + u_{xx}^2) \, dx \right]^{1/2} \tag{12}$$

is invariant under $u_t = v$, $v_t = -u_{xxxx}$. Hence it gives a satisfactory C_0-semigroup interpretation, through (2)-(3′), on the Banach space of all $(u(x), v(x)) = \mathbf{w}(x)$ for which (12) is finite. Again, if (f, g) denotes the Fourier transform (in X^*) of (u, v), then (12) equals the *Fourier transform norm*

$$\| w \| = \left[(\tfrac{1}{4}\pi) \int_\Omega (g^2 + q^4 f^2) \, dq \right]^{1/2}. \tag{12′}$$

In solving Cauchy problems, it may be appropriate to use Banach spaces which are not Hilbert spaces at all. Thus Hille [1, Chap. XX] constructed C_0-semigroups of linear transformations for the heat and telegrapher's equation in one space dimension (with $\Omega = R$), taking for $\mathscr{B}$ the spaces $C(R)$ or $L_p(R)$ and $C^1(R)$, respectively, as well as for the one-dimensional wave equation (using $C^1(R)$). But he did not solve the general problem, and his solution [1, p. 396] for the wave equation in R^r $(r > 1)$ in the space

[9] Actually, it is only a pseudonorm if $X = C^r$ is compact, since then $\| 1 \| = 0$.

$C^2(X)$ was incorrect. This is because initial conditions of class $\mathscr{C}^2$ at time $t = 0$ may create, by "focussing" [3, pp. 673-4], functions not of class $\mathscr{C}^2$ for $t > 0$, if $r > 1$. See also [14].

Pursuing Hille's ideas, T. W. Mullikin and the author showed in [2] how to construct a C_0-semigroup for *any* regular (i.e., "well-set") Cauchy problem (1). However, their construction does not give the natural "energy" norm in Example 4, and so it seems desirable to generalize it, and to explain it more clearly. This will be done below.

VI. Why Banach Spaces?

There is nothing sacrosanct about Banach spaces, of course, and one may well ask: why use Banach spaces at all?

The author knows of no unequivocal answer to this question, but would like to stress his profound agreement with Poincare's dictum [8, p. 23] that "it is physical applications which show us the important problems we have to set, and that again physics foreshadows the solutions." From this standpoint, Banach spaces seem adequate for treating physical systems having finite total energy—but not for those having infinite total energy,[10] like homogeneous turbulent flows. They also have the disadvantage, already explained, that one cannot use the same "standard" Banach space (like $L^2(X)$) for all problems.

Schwartz Distributions

As a "standard" function space, to be used for all Cauchy problems, the convex topological vector space $S(X)$ of Schwartz distributions is very attractive. Thus, it seems to be true that if $X = C^r$ is compact (a torus), then the system (1) defines a well-set[11] Cauchy problem in $S(X)$ if and only if (1) is regular. Again, every "hyperbolic" system (1) is regular, and defines a well-set Cauchy problem in $S(X)$, whether X is compact or not [4, Chap 7].

However, the Cauchy problem for the "parabolic" heat equation $u_t = u_{xx}$ is not well-set in $S(R)$. Moreover, in $S(X)$, the convergence of a sequence of functions u_m to u implies that of the partial derivatives $\partial u_m/\partial x_i$ to $\partial u/\partial x_i$: *differentiation is continuous*, as a linear operator on $S(X)$, and all Schwartz distributions are infinitely "differentiable." Indeed, the very smoothness of general statements in $S(X)$ reminds one of those made about the space of

[10] Function spaces suitable for treating systems having infinite total energy will be studied elsewhere, in a joint paper with Prof. Kampé de Fériet.

[11] Note that our definition of a "well-set" problem, adapted from Hille, is *not* that of a "correctly posed" problem made in [4, p. 218]: the latter uses *two* function spaces.

analytic functions in the Cauchy-Kowalewski Theorem—statements which finally turned out to be misleading [4].

In this connection, one wonders about the meaning of "convergence to 0" in the following example[12].

Example 6. Consider the "regular" hyperbolic system $u_t = v$, $v_t = u_{xx}$ obtained from the wave equation $u_{tt} = u_{xx}$ by the usual recipe. Define

$$F_m(x, 0) = \begin{cases} (1 - m^2x^2)/m, & |x| < 1/n, \\ 0 & \text{elsewhere}, \end{cases} \tag{13}$$

and consider the sequence of initial values

$$u_m(x, 0) = 2F_m(x, 0), \qquad v_m(x, 0) = 0. \tag{13$'$}$$

The solution of the system $u_t = v$, $v_t = u_{xx}$ for the initial values (13$'$), given by d'Alembert's formula, is

$$\begin{aligned} u_m(x, t) &= F_m(x + t) + F_n(x - t) \\ v_m(x, t) &= F_m'(x + t) - F_n'(x - t). \end{aligned} \tag{13$''$}$$

In $S(X)$, $(u_m, v_m) \to 0$; yet *physically*, the solutions all carry the same finite amount of (mechanical) *energy*. What if this "mechanical" energy were converted into heat by the *nonlinear* system

$$u_{tt} + |u_t|^2 = u_{xx}, \qquad w_t = w_{xx} + |u_t|^2?$$

The Space $K(X^*)$

If one is interested primarily in constructing function spaces giving rise to smooth general existence, uniqueness and continuity theorems for systems (1), then one should consider the normed vector space $K(X^*)$ of all *integrable* functions on X^* which have compact support, in the L_1-norm. Under (2$'$), $K(X^*)$ is mapped onto a subspace $A^*(X)$ of the space of all *analytic* functions on X. This subspace can therefore be given the same norm, and it is relevant that *every* Cauchy problem (1) is well-set in $A^*(X)$ as so constructed.

We reject this $A^*(X)$ on mathematical grounds because it is not complete (*not* a Banach space). It is also *physically* unreasonable: equations like the backwards heat equation $u_t = -u_{xx}$ (Example 1) are *not* well-set physically[13]. The author regards this agreement as supporting Hille's idea that Banach spaces are suitable spaces in which to treat well-set Cauchy problems.

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[12] The author wishes to acknowledge valuable comments by Prof. Avner Friedman and Dr. Bruce Kellogg on the statements made here.

[13] See J. C. Maxwell, "Theory of Heat," 10th ed., pp. 264-5, 1891.

VII. Diagonalizable Cauchy Problems

Actually, any well-set Cauchy problem (1) can be interpreted by a C_0-semigroup on a suitable *Hilbert space*, so far as solutions which correspond to physical states of finite total "energy" are concerned. This is easily shown when the $\lambda_l(\mathbf{q})$ are distinct for all $\mathbf{q}$—or, as in the wave equation, when they are distinct for almost all $\mathbf{q}$, provided eigenfunctions having distinct eigenvalues do not coalesce when eigenvalues cross. Such Cauchy problems may be called *diagonalizable*; they include the case of Hermitian $P(i\mathbf{q})$.

For diagonalizable *regular* systems (1), one can generalize the construction (10)-(11) as follows. Choose a basis of (unit) eigenvectors $\mathbf{b}_l(\mathbf{q})$ for each $\mathbf{q}$, whose components are *Borel functions* of $\mathbf{q}$. (This is always possible, since they are defined by algebraic equations.) Now let $\mathscr{B}$ be defined by the norm

$$\| f \| = \left\{ \sum_{l=1}^{n} \int_{\Omega^*} | \beta_l(\mathbf{q}) |^2 \, dQ \right\}^{1/2}, \tag{14}$$

for any initial data of the form

$$\mathbf{f}(\mathbf{q}, 0) = \sum \beta_l(\mathbf{q}, 0) \, \mathbf{b}_l(\mathbf{q}) \qquad \text{with} \qquad \| f \| < +\infty. \tag{15}$$

Since the solution for $t > 0$ is given by

$$\mathbf{f}(\mathbf{q}, t) = \sum \beta_l(\mathbf{q}, 0) \, e^{t\lambda_l(\mathbf{q})} \, \mathbf{b}_l(\mathbf{q}), \tag{15'}$$

an elementary calculation again gives (11′), and so (we omit many details, because they will be supplied for the general case below) the Cauchy problem for (1) defines a C_0-*semigroup*. Finally, it is evident that in the C_0-semigroup $\{T_t\}$ defined by (14)-(15)-(15′) the *norm* of T_t is

$$\| T_t \| = \sup_{x \neq \mathrm{O}} \| T_t[f] \| / \| f \| = e^{t\Lambda}. \tag{15}$$

Caution. Though the preceding norm defines a C_0-semigroup, it is *not unique*: any norm of the more general form

$$\| f \|_w = \left\{ \sum_{l=1}^{n} \int_{\Omega^*} C_l(\mathbf{q}) \, | \beta_l(\mathbf{q}) |^2 \, dQ \right\}^{1/2}, \qquad C_l(\mathbf{q}) > 0, \tag{16}$$

would define another Banach space $\mathscr{B}_w$ on which (1) also defines a C_0-semigroup. Indeed, as we will see in Section XI, such a norm may have the great advantage of ensuring that all solutions with finite initial $\| f \|_w$ actually satisfy (1) in the classical (literal) sense—something which is not often true otherwise.

VIII. Direct Integral Norms

We will now generalize the preceding construction, showing how to construct an energylike norm for *any* regular system (1), with respect to which (1) defines a C_0-semigroup on the Hilbert space of all "states" of finite "energy". More precisely, the *norm* is the *square-root* of an analog to physical *energy*.

The essential point is the dependence of the norm on the coefficients of (1): in general, one should *not* simply use a "standard" function space which, like $L_p(X)$ or $C^{(s)}(X)$, is determined by X alone. This is analogous to the situation in physical problems, where the mathematical expression for "energy" depends on the nature of the physical laws involved, and not just on the geometry of phase space (e.g., of (u, u_t)-space for the wave equation).

For any given $X = X_{r,s}$ and n, we first construct the linear space $B_n(X^*)$ of all vector fields $\mathbf{f}(\mathbf{q})$ on $X^* = R^s Z^{r-s}$ whose components $f_j(\mathbf{q})$ are *Borel functions* on $X^* = Q$, identifying vector fields which differ on a set of measure zero[14]. For each $\mathbf{q} \in X^*$, the *possible* $\mathbf{f}(\mathbf{q})$ form an n-dimensional vector space, in which any closed bounded convex symmetric neighborhood of $\mathbf{0}$ defines a norm $N(\mathbf{q}, \mathbf{f}(\mathbf{q}))$. For any selection of such norms which defines a Borel function $N(\mathbf{q}, \mathbf{f}(\mathbf{q}))$ of $\mathbf{q}$ and $\mathbf{f}$, the *direct integral norm*

$$\| f \|_{N,p}^{*} = \left[\int_{X^*} | N(\mathbf{q}, \mathbf{f}(\mathbf{q})) |^p \, dQ \right]^{1/p} \tag{17}$$

is defined, since any Borel function is measurable and the integrand is non-negative. Moreover, the $\mathbf{f}(\mathbf{q})$ having finite (N, p)-norm (17) form a Banach space $\mathscr{B}_{N,p}$.

We now show that one can always construct direct integral norms of the form (17) for any regular system (1) relative to which (3′) defines a C_0-semigroup. The proof will invoke the following basic result.

Theorem 3. *Let T be any linear transformation of $\mathscr{B}_{N,p}^{*}$ of the form*

$$T\left[\int_{X^*} \mathbf{f}(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ\right] = \int_{X^*} U(\mathbf{q})\, \mathbf{f}(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ,$$

where $U(\mathbf{q})$ is an $n \times n$ matrix-valued Borel function of $\mathbf{q}$. Then the (N, p)-norm of T is

$$M = \sup_{\mathbf{q},\mathbf{f}\neq 0} N(\mathbf{q}, U\mathbf{f})/N(\mathbf{q}, \mathbf{f}) = \sup_{\mathbf{q}\neq 0} M(\mathbf{q}). \tag{18}$$

Proof. By definition [9, p. 24], the (N, p)-norm of T is

$$\sup \left\{ \int_{X^*} N^p(\mathbf{q}, U\mathbf{f})\, dQ \Big/ \int_{X^*} N^p(\mathbf{q}, \mathbf{f})\, dQ \right\}^{1/p}.$$

[14] Instead of making this identification, we could select from each equivalence class the member $\mathbf{f}(\mathbf{q})$ which is the essential limes inferior of nearby values.

But this is bounded above by (18), and conversely can be approximated arbitrarily closely by it.

THEOREM 4. *Let replicas of a convex topological vector space $\mathscr{L}$ be made into a Banach space, for each $\mathbf{q} \in X^*$, by a norm $N(\mathbf{f}, \mathbf{q})$. On each replica $\mathscr{L}_{\mathbf{q}}$, let $\{T_{t,\mathbf{q}}\}$ define a C_0-semigroup; for any $\tau > 0$, let $\| T_{t,\mathbf{q}} \| \leqslant M(\tau) < +\infty$ on $0 < t < \tau$ uniformly in $\mathbf{q}$; and let $T_{t,\mathbf{q}}$ be a Borel function from $[0, +\infty)$ to $\mathscr{L}$. Define the direct integral p-norm on $\mathscr{L}^{X^*}$ by*

$$\| \mathbf{f}(\mathbf{q}) \| = \left\{ \int_{X^*} | N(\mathbf{f}, \mathbf{q}) |^p \, dQ \right\}^{1/p}. \tag{19}$$

Then the action of the $\{T_{t,\mathbf{q}}\}$ on the $\mathscr{L}_{\mathbf{q}}$ defines a unique C_0-semigroup on the Banach space $\mathscr{B}^*$ of those Borel $\mathbf{f}(\mathbf{q}) \in \mathscr{L}^{X^*}$ having finite norm (20).

SKETCH OF PROOF. It is obvious that (19) is a norm, and that $\mathscr{B}^*$ is complete under this norm; hence $\mathscr{B}^*$ is a Banach space. Moreoever, for any T, T_t maps $\mathscr{B}^*$ into itself linearly, with $\| T_t[f] \| \leqslant M(t) \| f \|$. Finally, for any fixed $f \in \mathscr{B}^*$, $\| T_t[f] - f \| \to 0$ as $t \downarrow 0$ by Lebesgue's dominated convergence theorem, which applies since $\| T_{t,\mathbf{q}} \|$ is bounded on $0 < t < \tau$.

Applying Theorems 3 and 4 to (3′) with *any* direct integral norm (17) on the space $\mathscr{B}^*$, we get the following corollary.

COROLLARY. *For (1) to generate a C_0-semigroup on $\mathscr{B}^*_{N,p}$, it is necessary and sufficient that*

$$M_{N,p}(P) = \sup_{\mathbf{q}, \mathbf{f} \neq 0} N(\mathbf{q}, \exp(tP(i\mathbf{q}))\, \mathbf{f}) / N(\mathbf{q}, \mathbf{f}) < +\infty, \qquad t > 0. \tag{19′}$$

REMARK. The T referred to in Theorem 3 are the linear transformations which commute with the translation operators S_a: $w(\mathbf{x}) \to w(\mathbf{x} + \mathbf{a})$. This is evident if X is compact, since the $\mathbf{f}(\mathbf{x}) = \mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}$ are characterized by the condition $S_{\mathbf{a}}(\mathbf{f}) = e^{i\mathbf{q}\cdot\mathbf{a}}\mathbf{f}$, which implies

$$S_{\mathbf{a}}(T\mathbf{f}) = T[S_{\mathbf{a}}\mathbf{f}] = T[e^{i\mathbf{q}\cdot\mathbf{a}}\mathbf{f}] = e^{i\mathbf{q}\cdot\mathbf{a}}T[\mathbf{f}],$$

whence $T[\mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}] = \mathbf{c}e^{i\mathbf{q}\cdot\mathbf{x}}$ for some $\mathbf{c} = U(\mathbf{q})\,\mathbf{b}$, depending *linearly* on T. We omit the details.

IX. STABILITY INDEX

In [2], one recipe was given for choosing $N(\mathbf{q}, \mathbf{b})$. We will now generalize this construction, first treating the finite-dimensional case corresponding to $P(i\mathbf{q})$ for fixed $\mathbf{q}$.

Finite-Dimensional Case

Let P be a linear operator on the n-dimensional complex vector space V; let $\rho(P)$ be the *spectral radius* of P. For any norm $N(\mathbf{f}) = \| \mathbf{f} \|_N$ on V, the *norm* of P is $\| P \|_N = \sup_{\mathbf{x} \neq \mathbf{o}} \| P\mathbf{f} \|_N / \| \mathbf{f} \|_N$, and it is well known that $\| P \|_N \geqslant \rho(P)$. Less well known is the fact that $\mathrm{Inf}_N \| P_N \| = \rho(P)$. However, this fact is also fairly obvious, since (for any $\epsilon \neq 0$) any linear operator P can be put in the *generalized* Jordan normal form whose irreducible blocks

$$\begin{pmatrix} \lambda & \epsilon & 0 & 0 \\ 0 & \lambda & \epsilon & 0 \\ 0 & 0 & \lambda & \epsilon \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

FIG. 1

have the form sketched in Fig. 1. For any norm of the form

$$N(\mathbf{f}) = \| \mathbf{f} \|_p = \left(\sum | f_i | p \right)^{1/p}, \qquad 1 \leqslant p \leqslant +\infty, \tag{20}$$

the norm of any matrix P in generalized Jordan normal form satisfies

$$\rho(P) \leqslant \| P \|_N \leqslant \rho(P) + \epsilon, \tag{20'}$$

whence the result is obvious.

Next, we consider P as an *infinitesimal operator* on V, associated as in (3′) with the vector differential equation

$$d\mathbf{f}/dt = P\mathbf{f}. \tag{21}$$

If the eigenvalues of P are $\lambda_1, \cdots, \lambda_n$, so that $\rho(P) = \sup | \lambda_i |$, then we define the *stability index* of P as (cf. Section III):

$$\nu(P) = \sup \mathrm{Re} \{\lambda_i\}. \tag{22}$$

It is well known (cf. (11′)) that $\rho(e^{tP}) = e^{t\nu(P)}$. Furthermore, $\nu(P)$ is the *type* of the C_0-semigroup e^{tP}, defined for any norm N [9, p. 306] as the right side of

$$\nu(P) = \lim_{t \to +\infty} \frac{1}{t} \ln (\| e^{tP} \|_N), \qquad P \text{ any matrix.} \tag{22'}$$

The preceding limit exists because $\ln \| e^{tP} \|_N$ is *subadditive* for $t > 0$ (see [9, Chap. VII]). Formula (22′) is the analog for the stability index of the well-known formula $\rho(P) = \lim_{m \to +\infty} \| P^m \|_N^{1/m}$, which is also valid for any norm N on a finite-dimensional vector space.

The preceding relation between "stability index" and "type" will now be extended to the direct integral norm (17).

Theorem 4 and its Corollary show that for (1) to generate a C_0-semigroup on $\mathscr{B}^*_{N,p}$, it is necessary and sufficient that the $N(\mathbf{q})$ be so chosen that the $e^{tP(i\mathbf{q})}$ have *uniformly bounded* norm on the r-dimensional space of all $\mathbf{b} = \mathbf{b}(\mathbf{q})$, relative to $N(\mathbf{q})$. But if $\Lambda(\mathbf{q}) = \sup_l \operatorname{Re} \{\lambda_l(\mathbf{q})\}$ is the *stability index* of $P(i\mathbf{q})$, then $\exp \Lambda(\mathbf{q})$ is the *spectral radius* of $\exp (tP(i\mathbf{q}))$.

We now show that the regularity of (1) is sufficient (as well as necessary, cf. Theorem 2) for it to correspond to a C_0-semigroup in a suitable $B^*_{N,p}$. To construct such a C_0-semigroup from (1), we generalize the direct integral (Fourier transform) norms, used in [2].

Definition. A *Jordan canonical norm* for a given system (1) is a direct integral norm (17) in which, for each $\mathbf{q}$

$$N^p(\mathbf{q}, \mathbf{f}) = \Sigma \,|\, b_i \,|^p, \qquad \mathbf{f} = \Sigma\, b_j \mathbf{e}_j\, t \tag{23}$$

the basis $\{\mathbf{e}_j\}$ being such that $P(i\mathbf{q})$ assumes a generalized Jordan canonical form with off-diagonal elements $\epsilon(\mathbf{q})$.

It will be shown in another paper[15] a nonsingular matrix *Borel function* $\| B_{jk}(\mathbf{q}) \|$ expressing the linear transformation $b_j(\mathbf{q}) = \Sigma\, B_{jk}(\mathbf{q}) f_k(\mathbf{q})$ which relates the f_k of (3) to the b_j of (23). This will assure that the matrix P can be reduced to generalized Jordan canonical form $J = BPB^{-1}$ with given Borel off-diagonal $\epsilon(\mathbf{q})$.

Lemma. *Given $p \geqslant 1$ and a positive integer m,*

$$\sup_{0<u<\to\infty} \{(1/u) \ln [1 + u^p + \cdots + u^{(m-1)p}/(m-1)!]^{1/p}\} = K_{m,p} \tag{24}$$

is a finite positive constant.

Proof. For $p = 1$, the expression in square brackets is just a truncated Taylor series of e^u, and so $K_{m,1} = 1$ (the supremum is approached as $u \downarrow 0$). For any $p > 1$, the continuous positive function in curly brackets tends to zero as $u \downarrow 0$ and also (since $\ln [\cdots] \sim (m-1)\, p \ln u$) as $u \uparrow +\infty$. Hence it assumes a finite maximum in between. Finally, in the limit as $p \uparrow \infty$, $[1 + u^p + \cdots + u^{(m-1)p}/(m-1)!]^{1/p}$ approaches the largest of the numbers $u^k/k!$ for $k = 1, \cdots, m-1$. Hence

$$K_{m,\infty} = \max_{1\leqslant 1 \leqslant m-1} \sup \left\{ \frac{1}{u} \ln \frac{u^k}{k!} \right\} = e[(m-1)!^{1/m-1}].$$

It would be interesting to evaluate $K_{m,2}$ explicitly.

[15] "Borel reductions to canonical form," by Kirby Baker, to appear in another issue of this journal.

COROLLARY 1. *If P is the $m \times m$ matrix of Fig. 1, and $N(\mathbf{b})$ is the norm (23), then*

$$\sup_{0<t<+\infty} (1/t) \ln \| e^{tP} \|_N \leqslant \lambda + m \epsilon K_{m,p} . \tag{25}$$

PROOF. Trivially, $\ln \| e^{tP} \|_N = \lambda t + \ln \| e^{tQ} \|_N$, where $Q = P - \lambda I$; hence it suffices to consider the case $\lambda = 0$. Moreover, by the triangle inequality applied to m-vectors,

$$\| e^{tQ}\mathbf{b} \| / \| \mathbf{b} \| \leqslant m \max \| e^{tQ} b_j \mathbf{e}_j \| / \| b \|,$$

where $\| b \| \geqslant | b_j |$. Hence $\| e^{tQ} \| \leqslant m \max \| e^{tQ} \mathbf{e}_j \|$, where the maximum is obviously assumed for $j = m$, giving

$$\| e^{tQ} \| \leqslant \frac{m}{t} \ln \left[1 + (t\epsilon)^p + \cdots + \frac{(t\epsilon)^{(m-1)p}}{(m-1)!} \right]^{1/p} . \tag{25'}$$

Setting $u = t\epsilon$ and $m/t = m\epsilon/u$, we get the desired conclusion as a corollary of (24).

Inspecting (25′) somewhat more carefully, we see that for any $\eta > 0$, we can choose T so large that if $t > T$ then either $\epsilon(\mathbf{q}) \leqslant \sqrt{\eta}$ or $u \geqslant T\sqrt{\eta}$ is arbitrarily large. Specifically, given $\eta > 0$, $\| e^{tQ} \| \leqslant m \sqrt{\eta}\, C_{m,p}$ for all $\mathbf{q}$ with $\epsilon(q) \leqslant \sqrt{\eta}$, regardless of T. Moreover we can choose $T(\eta)$ so large that $u \geqslant \sqrt{\eta}\, T(\eta)$ implies

$$u^{-1} \left\{ \ln \left[1 + u^p + \cdots + \frac{u^{(m-1)p}}{(m-1)!} \right]^{1/p} \right\} \leqslant m \sqrt{\eta}\, C_{m,p} .$$

Substituting above, since the limit is *uniform*, we obtain

COROLLARY 2. *For any Jordan canonical norm (23),*

$$\lim_{t \to +\infty} \frac{1}{t} \ln (\| e^{tP} \|) = \sup_{\mathbf{q}} \mathrm{Re}\, \{\lambda_l(\mathbf{q})\} = \Lambda(P). \tag{26}$$

Since the left side of (26) is the "type" [9, p. 306] of the C_0-semigroup, we have proved

THEOREM 5. *Any regular system (1) is realized, relative to any Jordan canonical norm with bounded $\epsilon(\mathbf{q})$, by a C_0-semigroup of type $\Lambda(P)$, where $\Lambda(P)$ is the stability index of (1).*

X. Admissible and Scalar-Equivalent Norms

The results of Sections VIII and IX refer to Banach spaces $\mathscr{B}^*_{N,p}$ defined by norms on the linear space of formal Fourier transforms (2) of solutions of (1). But we really want theorems about functions on X. To deduce such theorems from the results of Sections VIII and IX, one desires two new concepts.

Definition. A direct integral norm (17) is *admissible* when, for any Borel function $\mathbf{f}(\mathbf{q})$ of finite norm, (2′) defines a function $\mathbf{u}(\mathbf{x}) \in L_p(X)$. Two such norms $N(\mathbf{q}, \mathbf{f})$ and $N_1(\mathbf{q}, \mathbf{f})$ are *scalar-equivalent* when, for some *positive* (scalar) Borel function $C(\mathbf{q})$:

$$N_1(\mathbf{q}, \mathbf{f}) = C(\mathbf{q})\, N(\mathbf{q}, \mathbf{f}), \qquad \text{for all } \mathbf{q}. \tag{27}$$

For any admissible norm, we define $\mathscr{B}_{N,p}$ to consist of the vector fields $\mathbf{w}(\mathbf{q})$ defined as in (2′) by

$$\mathbf{w}(\mathbf{x}) = (2\pi)^{-r} \int_{\Omega^*} \mathbf{f}(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ \tag{28}$$

from $\mathbf{f}(\mathbf{q}) \in \mathscr{B}^*_{N,p}$. Relative to the Fourier transform norm

$$\| w \|_{N,p} \equiv \| f \|_{N,p}, \tag{28'}$$

the set of resulting $\mathbf{w}(\mathbf{x})$ is a Banach space (and, in fact, an abstract L_p-space).

The choice $p = 2$ and $N(\mathbf{q}, \mathbf{f}(\mathbf{q})) = (2\pi)^{r/2} \,|\, \mathbf{f} \,|$ in (17) evidently gives $L_2(X)$ as a special case. And indeed, the choice $p = 2$ is the most satisfactory one in many ways.

As an immediate consequence of Theorem 3, we have

Lemma 1. *Let T be any linear transformation of $\mathscr{B}^*_{N,p}$ of the kind defined in Theorem 3. Then the (N, p)-norm of T and the (N_1, p)-norm of T are the same.*

On the other hand, all norms on any finite-dimensional vector space are comparable. Hence, given any p (with $1 \leqslant p \leqslant +\infty$), $\mathbf{q}$, and $N(\mathbf{q}, \mathbf{f})$, one can always find a $c(\mathbf{q}) > 0$ such that

$$N_1(\mathbf{q}, \mathbf{f}) = c(\mathbf{q})\, N(\mathbf{q}, \mathbf{f}) \geqslant \left(\sum f_i^p\right)^{1/p}. \tag{29}$$

The resulting $N_1(\mathbf{q}, \mathbf{f})$ can be used to prove

LEMMA 2. *Let (1) define a C_0-semigroup on $\mathscr{B}^*_{N,p}$ for the norm $N(\mathbf{q}, \mathbf{f})$. Then there exists a scalar-equivalent norm $N_1(\mathbf{q}, \mathbf{f})$ such that (1) defines a C_0-semigroup on $\mathscr{B}_{N_1,p} \subset L_{p'}(\Omega)$ having the same $\| T(t) \|$. Here $p' = p/(p-1)$.*

To establish Lemma 2, it suffices to appeal to the generalized Plancherel Theorem [15, vol. 2, p. 254]. The norm in $\mathscr{B}_{N_1,p}$ is *at least* that in $L_{p'}(X)$—equality holding (up to a normalizing factor) if and only if $p = p' = 2$.

XI. DIFFERENTIABILITY

Finally, we want to show that a norm can be constructed, relative to which all $\mathbf{f}(\mathbf{q})$ of finite norm correspond to actual ("strong") solutions of (1). For this purpose, the following variant of Lebesgue's Dominated Convergence Theorem is useful.

LEMMA. *Let $F(\mathbf{q}; h)$ depend on a scalar parameter h, let $| F(q; h) | \leqslant U(\mathbf{q})$ for all h, where U is integrable over $X^* = Q$, and let $F(\mathbf{q}, h) \to F(\mathbf{q})$ as $h \to 0$. Then*

$$\operatorname*{Lim}_{h\to 0} \int_Q F(\mathbf{q}, h)\, e^{i\mathbf{q}.\mathbf{x}}\, dQ = \int_Q F(\mathbf{q})\, e^{i\mathbf{q}.\mathbf{x}}\, dQ. \tag{30}$$

We omit the proof.

COROLLARY 1. *If*

$$\int | f(\mathbf{q}) | \cdot | q_j |^k \, dQ < +\infty \qquad (k = 0, 1),$$

then

$$u(\mathbf{x}) = \int f(\mathbf{q})\, e^{i\mathbf{q}.\mathbf{x}}\, dQ$$

is in class $\mathscr{C}^1$ and, for $j = 1, \cdots r$,

$$\frac{\partial u}{\partial x_j}(\mathbf{x}) = \int q_j f(\mathbf{q})\, e^{i\mathbf{q}.\mathbf{x}}\, dQ. \tag{31}$$

To prove this result, we consider the difference quotient

$$\frac{e^{iq_jh} - 1}{h} = iq_j \frac{\sin q_jh}{q_jh} - hq_j^2 \frac{\sin^2 (q_jh/2)}{(q_jh)^2}. \tag{32}$$

Its absolute value is bounded by $| q_j |$, as is evident from complex geometry; moreover it tends to the limit iq_j as $h \to 0$.

$$\frac{1}{h}[u(\mathbf{x} + h\epsilon_j) - u(\mathbf{x})] = \int \left(\frac{e^{iq_jh} - 1}{h}\right) f(\mathbf{q})\, e^{i\mathbf{q}.\mathbf{x}}\, dQ$$

converges as $h \to 0$ to the integral on the right side of (31).

COROLLARY 2. *In order that $u \in \mathscr{C}^m$, it is sufficient that $k_1 + \cdots + k_r \leqslant m$ (k_j nonnegative integers) imply*

$$\int |f(\mathbf{q})| \cdot \prod |q_j|^{k_j} \, dQ < +\infty. \tag{33}$$

Literally Well-Set Cauchy Problems

The preceding result gives an easy sufficient condition for a direct integral norm with $p = 1$ to define a C_0-semigroup whose orbits consist of *literal* solutions of a first-order system $\partial u/\partial t = L[u]$. Namely, one first constructs a *suitable* Jordan canonical *norm* $N(\mathbf{q}, \mathbf{f})$ (giving a C_0-semigroup), and then looks for a scalar-equivalent norm $N_1(\mathbf{q}, \mathbf{f})$ such that the boundedness of the integral in (17) implies that of all integrals (33) for which $\partial^m/\partial q_1^{k_1} \cdots \partial q_r^{k_r}$ occurs on the right side of [16, (1)].

This will guarantee that $P(i\mathbf{q})\,\mathbf{f}(\mathbf{q})$ is (absolutely) *integrable* for all $\mathbf{f}$ with $\| \mathbf{f}(\mathbf{q}) \|_{N,1} < \infty$.

It follows by (3′) that, on any orbit $\mathbf{f}(\mathbf{q}, t)$ in the Fourier transform space:

$$\begin{aligned} \frac{1}{\Delta t}[\mathbf{f}(\mathbf{q}, t + \Delta t) - \mathbf{f}(\mathbf{q}, t)] &= \frac{1}{\Delta t}[e^{\Delta t P(i\mathbf{q})} - I)\,\mathbf{f}(\mathbf{q}, t) \\ &= EQ(\Delta t P(i\mathbf{q}))\,[P(i\mathbf{q})\,\mathbf{f}(\mathbf{q}, t)], \end{aligned} \tag{34}$$

where

$$EQ(z) = \sum_0^\infty [z^k/(k+1)!] = (e^z - 1)/(z).$$

But now, a direct calculation shows that, relative to the generalized Jordan canonical norm of Section VIII, the Euclidean norm of $EQ(\Delta t P(i\mathbf{q}))$ is at most $\exp[\Delta t(\Lambda(P) + \epsilon)]$. Hence, $N_1(\mathbf{q}, \mathbf{f})$ having been chosen so that $P(i\mathbf{q})\,\mathbf{f}(\mathbf{q})$ is absolutely integrable for all $f \in \mathscr{B}_{N_1,1}$, the conditions for (30) are again fulfilled. Therefore $\partial \mathbf{u}/\partial t$ exists, is continuous, and (since, for each fixed $\mathbf{q}$, $EQ(\Delta t P i\mathbf{q}) \to I$ as $\Delta t \to 0$) satisfies $\partial \mathbf{u}/\partial t = L[\mathbf{u}]$. We have thus proved.

THEOREM 6. *Given any regular system (1), there exists a Jordan canonical norm (17), (23) relative to which (1) induces a C_0-semigroup on the Banach space $\mathscr{B}_{N,2}$, whose orbits are all literal solutions of (1).*

XII. THE INFINITESIMAL GENERATOR

So far, the infinitesimal generator P of the C_0-semigroup of bounded linear operators e^{tP} on $\mathscr{B} = \mathscr{B}_{N,p}$ has been treated purely symbolically. In fact, it is an (unbounded, closed) linear operator on $\mathscr{B}$. As such, P can be defined directly from (1) and (2)-(2′), setting

$$P[u] = (2\pi)^{-r} \int_Q P(i\mathbf{q})\, \mathbf{f}(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ. \tag{35}$$

If $P(i\mathbf{q})$ has for each $\mathbf{q}$ a basis of eigenvectors $\mathbf{b}_l(\mathbf{q})$ with eigenvalues $\lambda_l(\mathbf{q})$, then, in the associated Jordan canonical norm, $P[u] \in \mathscr{B}$ for

$$\mathbf{u}(\mathbf{x}) = \int \sum \beta_l(\mathbf{q})\, \mathbf{b}_l(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ, \tag{36}$$

if and only if

$$\left[\int \sum |\lambda_l(\mathbf{q})\, \beta_l(\mathbf{q})|^p\, dQ\right]^{1/p} < +\infty. \tag{36′}$$

Alternatively, one can define P in terms of the general theory of semigroups [9, p. 302], as the closure of the limit

$$P[u] = \lim_{t\downarrow 0} t^{-1}(T_t[u] - u) \qquad \text{in } \mathscr{B} \tag{37}$$

whenever this limit exists. But, in terms of the notation already introduced above, this limit is

$$\lim_{t\downarrow 0} \sum [(e^{t\lambda l} - 1)/t]\, \beta_l(\mathbf{q})\, b_l(\mathbf{q}) = \sum \lambda_l(i\mathbf{q})\, \beta_l(\mathbf{q})\, \mathbf{b}_l(\mathbf{q}). \tag{37′}$$

Moreover, by Lebesgue's dominated convergence theorem, the limit exists whenever (36′) holds, since $t^{-1}(e^{t\lambda l} - 1) = O(1)$ if $\operatorname{Re}\{\lambda_l\} < \Lambda < +\infty$.

Using slightly more complicated formulas, one can snow that the preceding results hold even when $P(i\mathbf{q})$ is not similar to a diagonal matrix, but is reduced to Jordan canonical form. This argument proves

THEOREM 7. *In the representation of Sections VIII and IX, the differential operator P is the infinitesimal generator of the C_0-semigroup of the e^{tP} on $\mathscr{B}$.*

In any case, the domain of P always contains the dense subspace $K(X^*)$ of Section VI.

Further, P has a bounded resolvent for $\operatorname{Re}\{\lambda\} > \Lambda$ defined by [9, (10.6.3)], as:

$$(\lambda I - P)^{-1} = \int_0^\infty e^{-\lambda t}\, T_t[u]\, dt. \tag{38}$$

Moreover, since [9, p. 322, foot] any C_0-semigroup is of class $(0, A)$, one can reconstruct $\{T_t\}$ from P by using the generalization [9, (11.6.2)] of Cauchy's integral formula, which is in our notation

$$T_t[u] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda I - P)^{-1} u \, d\lambda, \qquad \gamma > \Lambda(P). \tag{39}$$

This shows in particular that P (which has the characteristic properties of the Feller-Miyadera-Phillips Theorem [9, Theorem 12.3.1]) defines a ***unique*** C_0-semigroup on $\mathscr{B}$.

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